Estimation of the parameters of a diffusion with discontinuous coefficients

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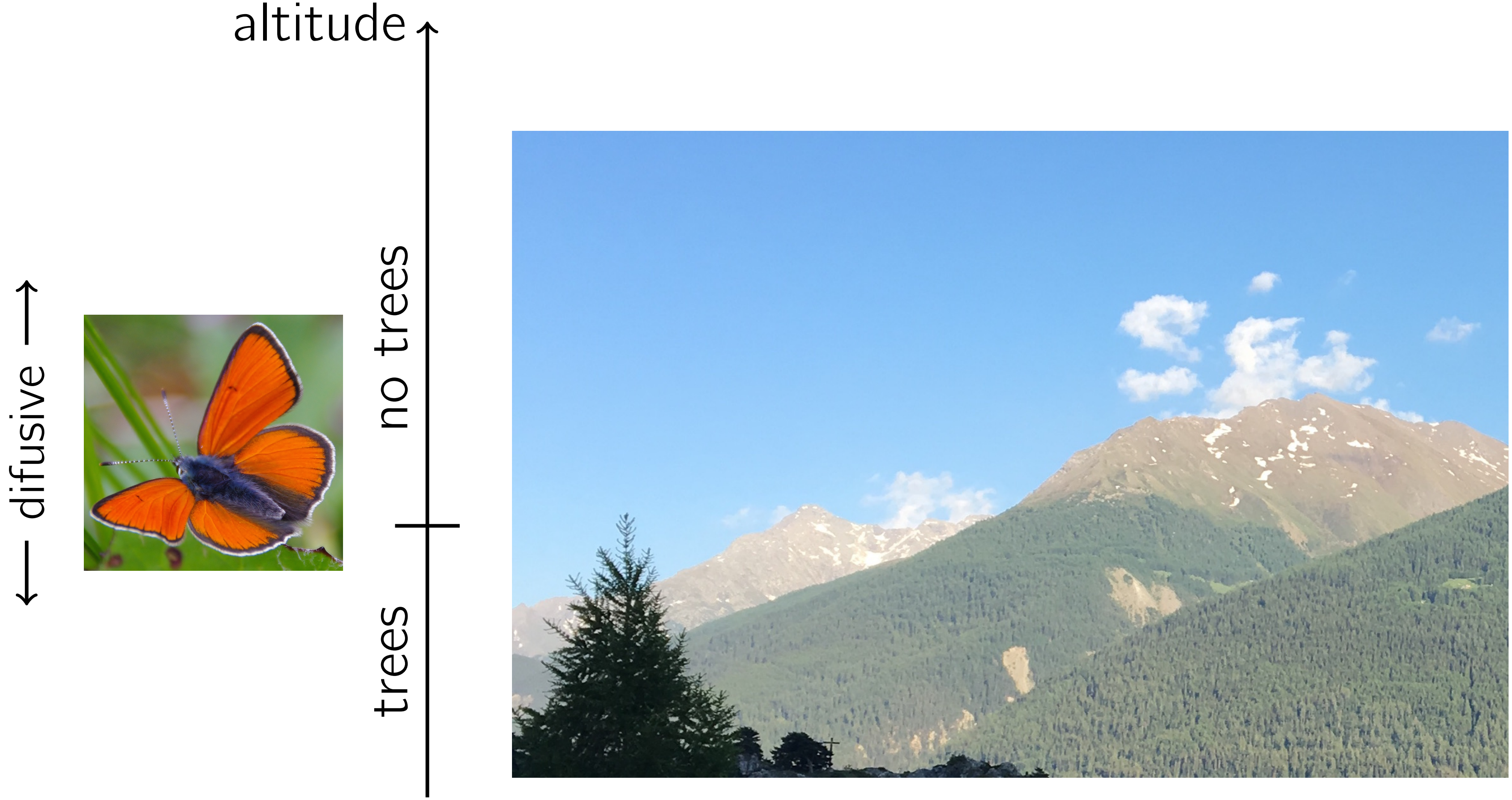
Journées de Probabilités
Aussois, June 2017

A (local) motivation



A warm-up: Case of a constant σ

- $\mathrm{d}X_t = \sigma\,\mathrm{d}B_t$ observed at high-frequency $\{X_{kT/n}\}_{k=0}^{n}$.
- A natural estimator is

$$\sigma_n^2 = \frac{1}{T}\sum_{i=1}^{n}(X_{iT/n} - X_{(i-1)T/n})^2.$$

- Itô formula $\Longrightarrow$

$$(X_t - X_s)^2 = 2\sigma\int_s^t (X_r - X_s)\,\mathrm{d}B_s + \sigma^2(t-s)$$

and then $T\sqrt{n}(\sigma_n^2 - \sigma^2) = \sqrt{n}M_T^n$ is a martingale.

Itô again $\Longrightarrow \forall t \geq 0$,

$$\langle\sqrt{n}M^n\rangle_t \xrightarrow[n\to\infty]{\mathbb{P}} 2\sigma^4 \text{ and } \langle\sqrt{n}M^n, B\rangle_t \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$

With a CLT on martingales,

$$\sqrt{n}(\sigma_n^2 - \sigma^2) \xrightarrow[n\to\infty]{\text{law}} \sqrt{2}\sigma^2 W_T/T,\ W \text{ BM indep. from } B.$$

The Oscillating Brownian motion (OBM)

Terminology of Keilson & Wellner (1978)

$$X_t = x + \int_0^t \sigma(X_s)\,\mathrm{d}B_s,\ \sigma(x) = \begin{cases} \sigma_+ & \text{if } x \geq 0 \\ \sigma_- & \text{if } x < 0. \end{cases}$$

- Strong existence, uniqueness ($\Longleftarrow$ Le Gall, 1978)
- Analytic formula of the density, occupation time (Keilson & Wellner, 1978)
- Convergence of the Euler scheme (Chan & Stramer, 1989, Yan, 2002, ...)
- Approximation by Random Walks (Keilson & Wellner, 1978; Helland, 1982; Étoré, 2006 ...)

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Applications

- In a continuous time version of the Self-Exciting Threshold Auto-Regressive (SETAR) models (Tong, 1983).
- In finance, it mimicks a leverage effect of log-prices.
- In population ecology, it models change of habitats.

The realized volatility

Observations: $X_{kT/n}$, $k = 0, \dots, n$ (high-frequency data)

How to estimate σ_+ and σ_-?

Realized volatility type estimators

$$\sigma_\pm(n)^2 = \frac{\sum_{k=1}^n (X^\pm_{kT/n} - X^\pm_{(k-1)T/n})^2}{\frac{T}{n}\sum_{k=1}^n \mathbb{1}_{\pm X_{kT/n}\geq 0}}$$

Idea Itô-Tanaka formula $\Longrightarrow$

$$X^\pm_t = X^\pm_0 + \sigma_\pm \int_0^t \mathbb{1}_{\pm X_s \geq 0}\,\mathrm{d}B_s + \frac{1}{2}L_t(X)$$

with

- $L_t(X)$ local time at 0 (finite variation process).
- $\left\langle \int_0^\cdot \mathbb{1}_{\pm X_s\geq 0}\,\mathrm{d}B_s \right\rangle_t = Q^\pm_t$ occupation time of $\mathbb{R}_\pm$

A convergence result

(i) $\sigma_\pm(n)$ is a consistent estimator of $\sigma_\pm$ as $n \to \infty$.

(ii) When $T = 1$,

$$\sqrt{n}(\sigma_\pm(n)^2 - \sigma_\pm^2)$$

$$\xrightarrow[n\to\infty]{\text{stable}} \frac{\sqrt{2}\sigma_\pm^2}{Q_1^\pm} \int_0^1 \mathbb{1}_{\pm X_s \geq 0}\, \mathrm{d}\widetilde{B}_s - \frac{1}{Q_1^\pm} \frac{2\sqrt{2}}{3\sqrt{\pi}} \frac{\sigma_-\sigma_+}{\sigma_- + \sigma_+} L_1(X),$$

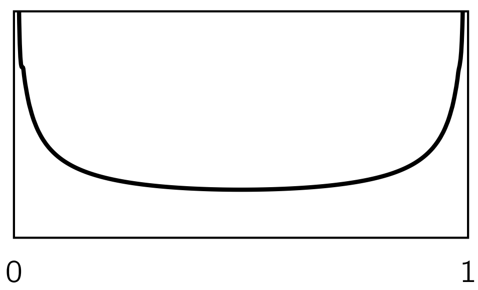
$\widetilde{B}$ is a BM indep. from B.

Remarks

- Joint convergence of $(\sigma_+(n), \sigma_-(n))$.
- Using Girsanov's theorem, we could consider the presence of drift (the limit laws are changed).
- By scaling, high-frequency estimation = long time estimation (not true in presence of drift).

Comments

• The limit depends on $Q_1^{\pm}$ which follows a law of ArcSine type



$\Longrightarrow$ either Q_1^+ or Q_1^- is likely to be close to 1
$\Longrightarrow$ either σ_+ or σ_- is likely to be loosely estimated.

• The process X is null recurrent
$\Longrightarrow$ the limit law is a mixture of normal distribution.

• There is an asymptotic bias which is due to the discontinuity.

Some ingredients of the proof

We have to prove, in particular, convergences of type

- $\sqrt{n}[L(B), L(B)] \xrightarrow[n\to\infty]{\text{proba}} \frac{4\sqrt{2}}{3\sqrt{\pi}} L(B)$
- $\sqrt{n}[L(X), X] \xrightarrow[n\to\infty]{\text{proba}} 0$
- $\sqrt{n}[L(X), |X|] \xrightarrow[n\to\infty]{\text{proba}} 0$

with $[Y, Z] = \sum_{i=1}^{n} (Y_{i/n} - Y_{(i-1)/n})(Z_{i/n} - Z_{(i-1)/n})$

For this, we use that for a suitably decreasing function f,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} f(\sqrt{n} X_{i/n}) \xrightarrow[n\to\infty]{\text{proba}} c(f) L_1(X)$$

by adapting some results of J. Jacod (1998) to the OBM by reducing it to a Skew Brownian motion $Y_t = B_t + \gamma L_t(Y)$. Computations are based on explicit expression of the density (this limits immediate generalizations).

Removing the asymptotic bias

Our estimator is changed to

$$\widehat{\sigma}_{\pm}(n)^2 = \frac{\sum_{k=1}^{n}(X^{\pm}_{k/n} - X^{\pm}_{(k-1)/n})(X_{k/n} - X_{(k-1)/n})}{\frac{1}{n}\sum_{k=1}^{n} \mathbb{1}_{X_{k/n}\geq 0}}$$

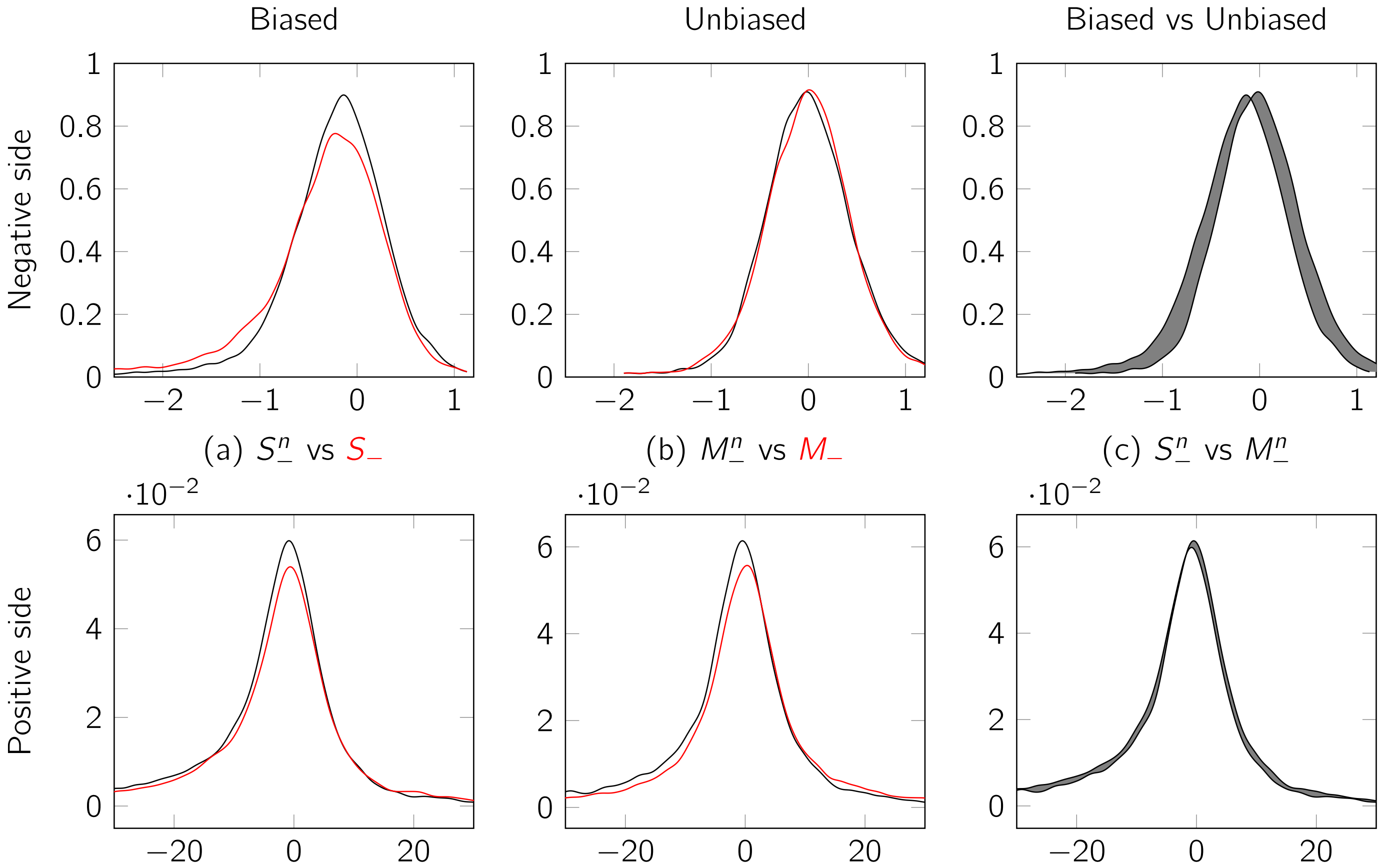
$$\sqrt{n}(\widehat{\sigma}_{\pm}(n)^2 - \sigma_{\pm})^2 \xrightarrow[n\to\infty]{\text{stable}} \frac{\sqrt{2}\sigma_{\pm}^2}{Q_1^{\pm}} \int_0^1 \mathbb{1}_{\pm X_s \geq 0}\, \mathrm{d}\widetilde{B}_s.$$

The reason is that

$$\sqrt{n}\sum_{k=1}^{n}(X^{+}_{k/n} - X^{+}_{(k-1)/n})(X^{-}_{k/n} - X^{-}_{(k-1)/n})$$

$$\xrightarrow[n\to\infty]{\text{stable}} \frac{2\sqrt{2}}{3\sqrt{\pi}} \frac{\sigma_{-}\sigma_{+}}{\sigma_{-}+\sigma_{+}} L_1(X).$$

Numerical illustration on $S^n_\pm := \sqrt{n}(\sigma_\pm(n) - \sigma)$ $M^n_\pm := \sqrt{n}(\widehat{\sigma}_\pm(n) - \sigma)$



(a) S^n_- vs S_- (b) M^n_- vs M_- (c) S^n_- vs M^n_-

(d) S^n_+ vs S_+ (e) M^n_+ vs M_+ (f) S^n_+ vs M^n_+

$\sigma_- = 1/2$, $\sigma_+ = 2$, $n = 500$, on 10 000 paths

Estimation of a two-valued drift

$$X_t = x + \int_0^t \sigma(X_s)\,\mathrm{d}B_s + \int_0^t b(X_s)\,\mathrm{d}s,$$

with

$$\sigma(x) = \begin{cases} \sigma_+ & \text{if } x \geq 0 \\ \sigma_- & \text{if } x < 0 \end{cases} \text{ and } b(x) = \begin{cases} b_+ & \text{if } x \geq 0 \\ b_- & \text{if } x < 0. \end{cases}$$

How to estimate (b_-, b_+)?

- We should consider long time estimation.
- The respective signs of b_+ and b_- are fundementals:

	$b_+ > 0$	$b_+ = 0$	$b_+ < 0$
$b_- > 0$	transient	null recurrent	ergodic
$b_- = 0$	transient	null recurrent	null recurrent
$b_- < 0$	transient	transient	transient

Estimation of a two-valued drift

Itô-Tanaka formula + Maximization of the Girsanov weight

$$\beta_\pm = \pm \frac{X_T^\pm - X_0^\pm - L_T/2}{Q_T^\pm} = b_\pm + \frac{M_t^\pm}{Q_T^\pm}$$

where $M^\pm = \int_0^\cdot \mathbb{1}_{\pm X_s \geq 0} \,\mathrm{d}B_s$, $\langle M^\pm \rangle = Q^\pm$.

$\Longrightarrow$ Empirical estimator for large T

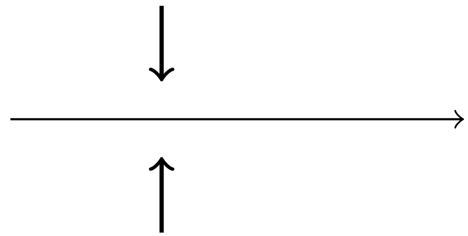
$$\widehat{b}_\pm = \pm \frac{X_T^\pm - X_0^\pm - \hat{L}_T/2}{\hat{Q}_T^\pm}$$

where $\hat{Q}^\pm$, $\hat{L}$ are empirical estimators of $Q^\pm$, L.

The convergence of the estimator depends on the long time behavior of $Q_T^\pm$, and of the regime of X.

Estimation of a two-valued drift

Ergodic case: $\begin{cases} b_+ < 0 \\ b_- > 0 \end{cases}$



- Unique invariant measure $\mu(\,\mathrm{d}x) \propto \frac{2}{\sigma(x)} \exp\left(-\int_0^x \frac{2b(y)}{\sigma(y)} \mathrm{d}y\right) \mathrm{d}x$
- Ergodicity and martingale CLT $\Longrightarrow$

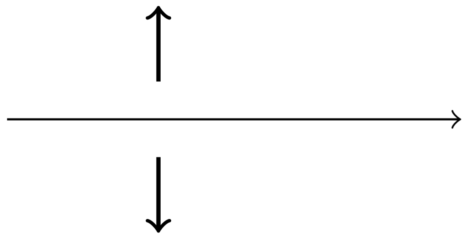
$$\frac{Q_T^\pm}{T} \xrightarrow[T\to\infty]{\text{a.s.}} c^\pm \text{ and } \frac{M_T^\pm}{\sqrt{T}} \xrightarrow[T\to\infty]{\text{law}} \sqrt{c^\pm} G^\pm$$

for (G^-, G^+) a Gaussian rv.

$\Rightarrow$ The estimator is consistent and $\beta_\pm$ converges to $b_\pm$ at rate $1/\sqrt{T}$ with a CLT.

Estimation of a two-valued drift

"Repulsive" case: $\begin{cases} b_+ > 0 \\ b_- < 0 \end{cases}$



- The process is transient and the last passage time to 0 is finite a.s.
- With probab. $p = \frac{\sigma_- b_+}{\sigma_- b_+ - \sigma_+ b_-}$, the process ends up in the positive axis (Watanabe, 1995).

$\Rightarrow$ Q_T^+/T converges to 1 and β_+ converges to b_+ at rate $1/\sqrt{T}$ with a CLT.

$\Rightarrow$ The estimator of b_- is meaningless.

- Or the symmetric situation holds.

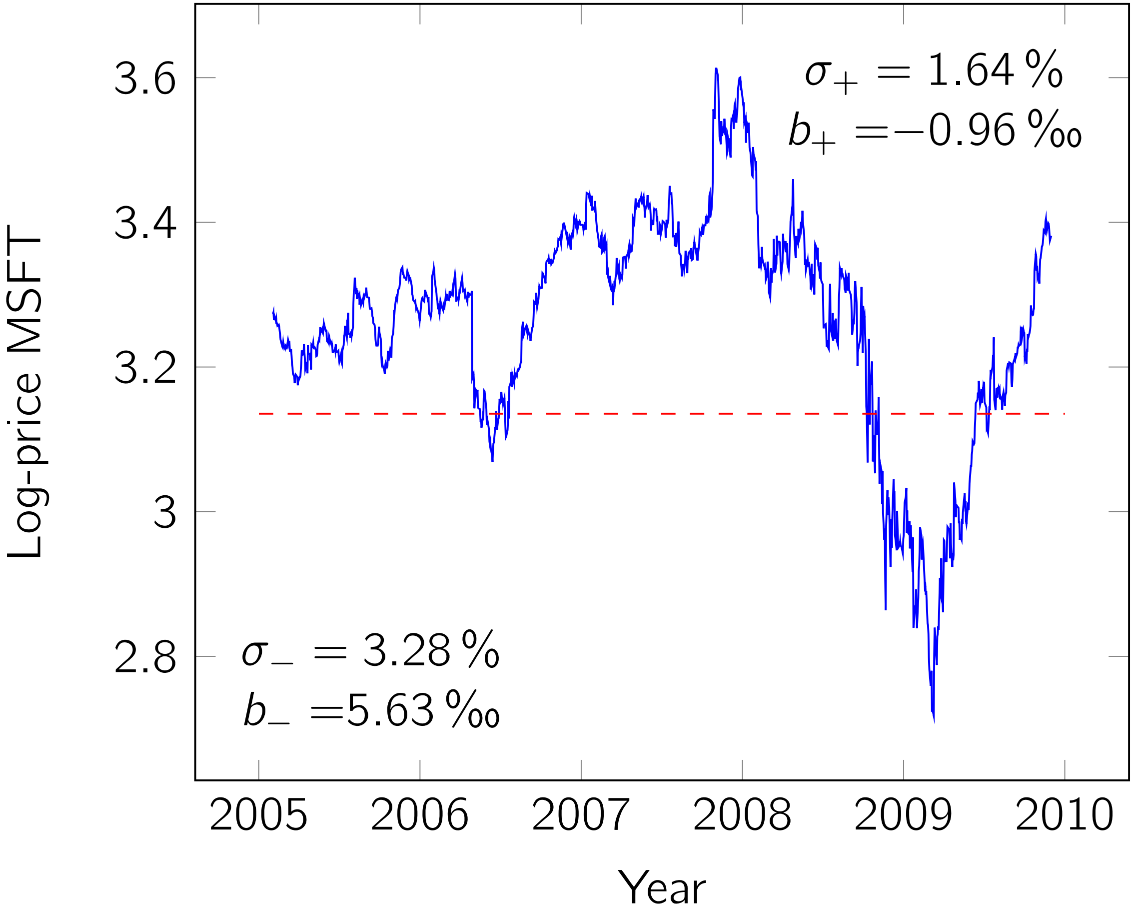
Other cases should be treated individually and may lead to other rates.

Application to financial data

- P. Mota & M. Esquível (2014) have proposed a continuous time version of a SETAR model with delay and threshold regime switching (DTRS).
- Their model uses an artifical thin layer for switching to avoid immediate switchings.
- They propose a least squares estimation procedure (coming from time series).
- 21 stocks are analyzed (2005-2010): leverage and mean-reverting effects hold for most of them.

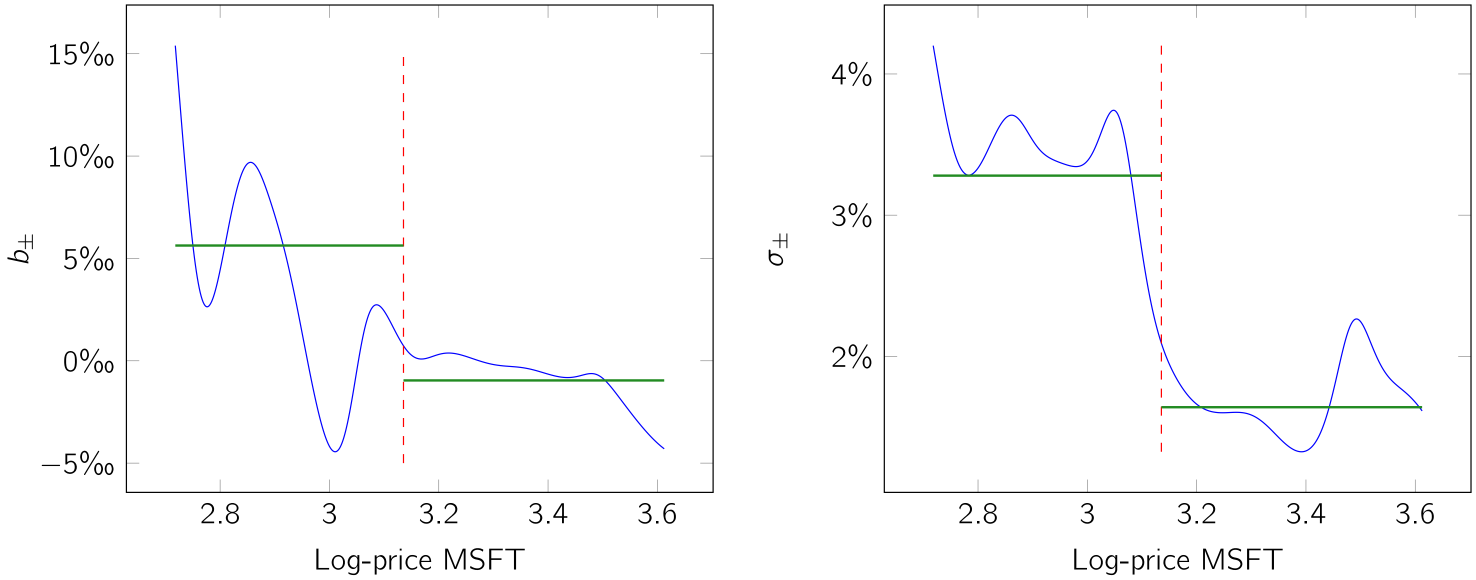
Our estimators gives consistent results with these ones.

Leverage and mean-reverting effects



The threshold is detected with the AIC model selection.

Comparison with a non-parametric estimator



Comparison with a Nadaraya-Watson non-parametric estimation.

Conclusion

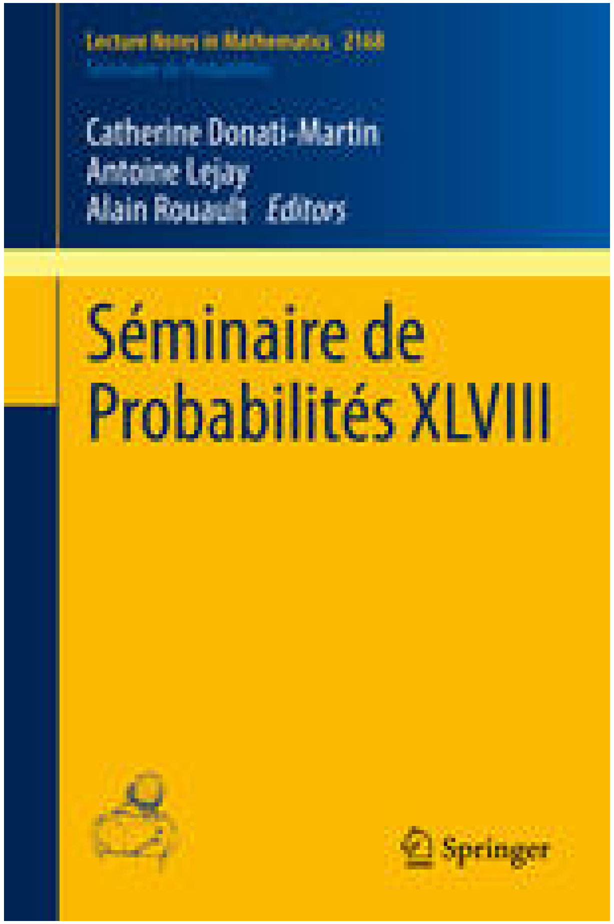
- The problem of estimation of SDE with discontinuous coefficients is surprisingly open.
- Asymptotics of occupation and local times play a very important role.
- Heavily relies on the limits theorems contained in the book Jacod & Protter. However, they should be adapted to the Skew Brownian motion (some questions are left open).
- The presence of a drift really changes the picture.

✠ AL & PP, Statistical estimation of the Oscillating Brownian Motion, arxiv:1701.02129 (2017).

✠ Estimation of drift, application to financial data: works in progress.

Et maintenant

une page de publicité...



- Crée en 1967, lié à l'École française de probabilités
- $\approx$ 1 volume/an aux LNM ($\sim$ 500 pages)
- Articles de synthèse
- De nombreux thèmes traités
- Site WEB avec base de données + notices

http://sites.mathdoc.fr/SemProba/