Persistent random walks: functional scaling limits

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Persistent Random Walk: a long range memory process

- Persistent Random Walk:

$$S_0 := 0, \quad S_n := \sum_{k=1}^{n} X_k, \quad n \geq 0,$$

where $\{X_k\}_{k \in \mathbb{Z}}$ is some stochastic process taking values in $\{-1, 1\}$.

- Distribution of jumps:

$$\mathcal{L}(X_{n+1} | X_k, k \leq n) := \mathcal{B}(p_{X_n, L_n}) = p_{X_n, L_n} \delta_{-1} + (1 - p_{X_n, L_n}) \delta_1.$$

with

  - $\{p_{(b,n)}\}_{(b,n) \in \{-1,1\}, \times \mathbb{N}^* \cup \{\infty\}}$ real numbers in $[0, 1]$;
  - $L_n :=$ card $\{\ell \geq 0 : X_n = X_{n-1} = \cdots = X_{n-\ell} \neq X_{n-\ell-1}\}$.

- Remark: the random variable L_n may be unbounded.

Objective: functional scaling limit

- Stating the functional convergence

$$\left\{\frac{S_{\lfloor ut\rfloor} - \mathbf{m}_S ut}{\lambda(u)}\right\}_{t\geq 0} \quad \text{or} \quad \left\{\frac{S_{ut} - \mathbf{m}_S ut}{\lambda(u)}\right\}_{t\geq 0} \quad \overset{\mathcal{D}}{\Longrightarrow} \quad \{Z(t)\}_{t\geq 0}.$$

- Remark: Since $|X_k| = 1$, the process $\{S_t\}_{t\geq 0}$ is ballistic or sub-ballistic. Consequently

$$\begin{cases} \mathbf{m}_S \in [-1, 1] \\ \lambda(u) \text{ increase at a rate at most linear.} \end{cases}$$

- Determination of $\mathbf{m}_S$, $\lambda(u)$, $\mathcal{D}$ and $\{Z(t)\}_{t\geq 0}$.

Outline

VLMC : Variable Length Markov Chain

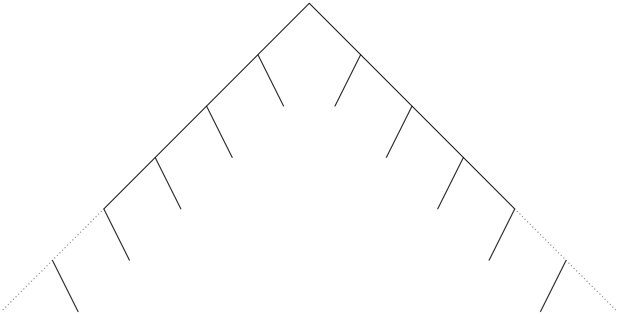

Figure: The infinite double comb context tree $\mathbb{T}$.

- $\mathcal{A} := \{\mathtt{d}, \mathtt{u}\}$ and $\mathcal{L} := \mathcal{A}^{-\mathbb{N}}$
- labels on edges, labels on leaves
- $\overleftarrow{\mathrm{pref}} : \mathcal{L} \to \overline{\mathcal{A}^{+}}$
- $q_c \in \mathcal{M}^1(\mathcal{A})$, $c \in \mathcal{C}$.

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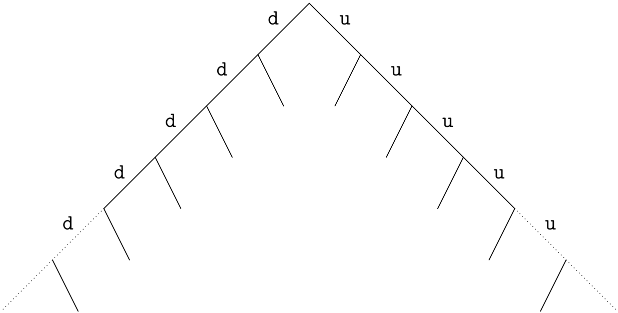

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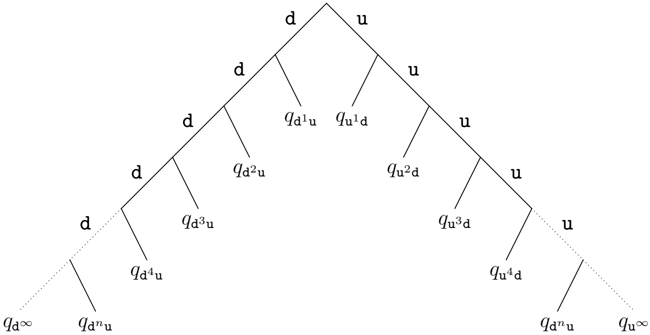

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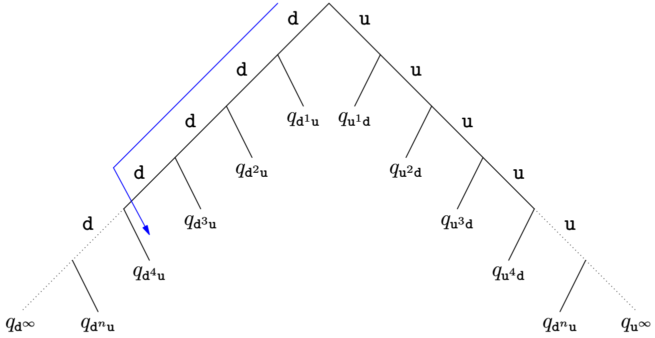

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Existence

Definition

A VLMC is a Markov chain $\{U_n\}_{n\geq 0}$ on the state space $\mathcal{L}$ with transitions given for $\ell \in \mathcal{A}$ by

$$\mathbf{P}(U_{n+1} = U_n\ell | U_n) = q_{\overleftarrow{\mathrm{pref}}\,(U_n)}(\ell).$$

Setting

$$X_k := \begin{cases} U_0^{(k)} & k \leq 0 \\ U_k^{(0)} & k > 0 \end{cases} \qquad \{-1, 1\} \cong \{\mathtt{d}, \mathtt{u}\},$$

it completely defines the Persistent Random Walk.

Assumption

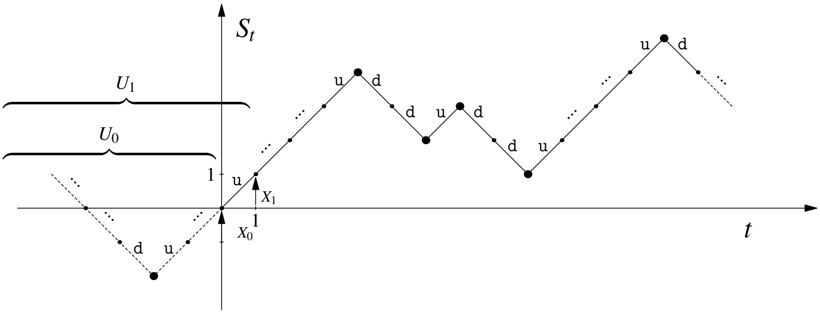

Figure: Sample path of a Persistent Random Walk.

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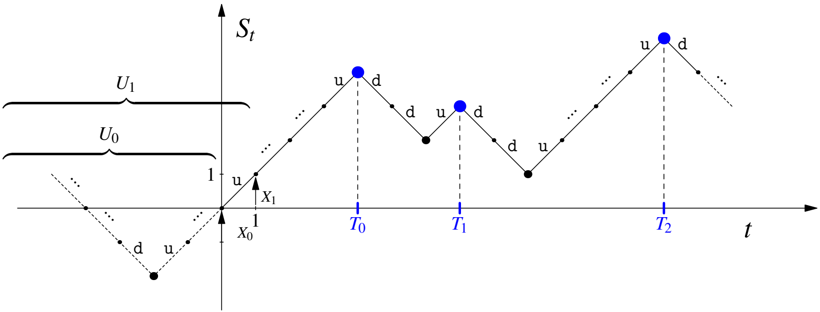

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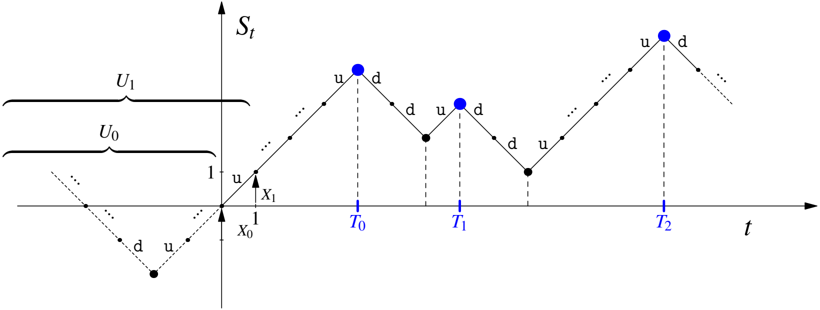

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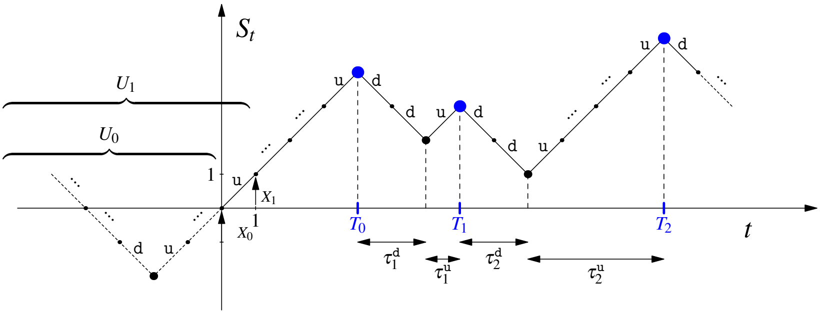

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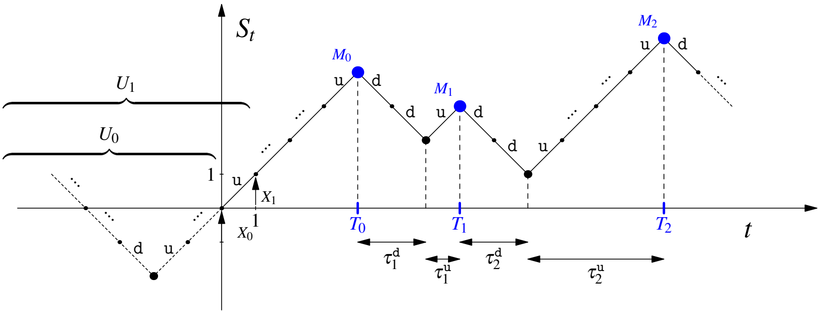

Figure: Sample path of a Persistent Random Walk.

Basics

Assumption

Assume $T_0 < \infty$, a.s. for almost all $U_0 \in \mathcal{L}$.

Proposition

For the double infinite comb VLMC and under the assumption on T_0, it holds

1. *for all $i \geq 0$, $T_i < \infty$ a.s. (and for almost all initial conditions)*
2. *$\{\tau_n^{\mathrm{d}}\}_{n\geq 0}$ and $\{\tau_n^{\mathrm{u}}\}_{n\geq 0}$ are independent sequences of i.i.d. $\mathbb{N}^*$-valued random variables.*

Remark: Under such assumptions, the persistence times L_n may stay unbounded.

Mean drift condition

Define the mean drift

$$\mathbf{m}_S := \begin{cases} \mathbf{b}_S & \text{if} \quad \mathbf{E}(\tau_1^{\mathtt{u}}) = \mathbf{E}(\tau_1^{\mathtt{d}}) = \infty; \\ \mathbf{d}_S := \frac{\mathbf{E}(\tau_1^{\mathtt{u}}) - \mathbf{E}(\tau_1^{\mathtt{d}})}{\mathbf{E}(\tau_1^{\mathtt{u}}) + \mathbf{E}(\tau_1^{\mathtt{d}})} & \text{otherwise,} \end{cases}$$

where

$$\mathbf{b}_S := \lim_{t\to\infty} \frac{\mathcal{T}_{\mathtt{u}}(t) - \mathcal{T}_{\mathtt{d}}(t)}{\mathcal{T}_{\mathtt{u}}(t) + \mathcal{T}_{\mathtt{d}}(t)}, \quad \mathcal{T}_\ell(t) := \mathbf{P}(\tau_1^\ell > t), \quad \ell \in \mathcal{A}.$$

Assumption

The mean drift $\mathbf{m}_S$ is well defined and not extremal, i.e. $\mathbf{m}_S \in (-1,1)$. Moreover, there exists $\alpha \in (0,2]$ such that

$$\tau_1^{\mathtt{c}} := (1 - \mathbf{m}_S)\tau_1^{\mathtt{u}} - (1 + \mathbf{m}_S)\tau_1^{\mathtt{d}} \in D(\alpha)$$

i.e. belongs to the domain of attraction of an α-stable distribution.

Normalizing functions

For $\ell \in \mathcal{A} = \{\mathtt{d}, \mathtt{u}\}$, define

1. truncated first moment: $\Theta_\ell(t) := \mathbf{E}[\tau_1^\ell \wedge t]$;
2. truncated second moment: $\mathbb{V}_\ell(t) := \mathbf{E}[(\tau_1^\ell)^2 \wedge t]$;

and introduce the quantities

$$\begin{cases} \Sigma^2(t) := (1 - \mathbf{m}_S)\mathbb{V}_{\mathtt{u}}\left(\frac{t}{1-\mathbf{m}_S}\right) + (1 + \mathbf{m}_S)\mathbb{V}_{\mathtt{d}}\left(\frac{t}{1+\mathbf{m}_S}\right) \\ \Theta(t) := \Theta_{\mathtt{u}}(t) + \Theta_{\mathtt{d}}(t). \end{cases}$$

Set $\lambda(u) := a \circ s(u)$ with

$$\begin{cases} a(u) := \inf\{t > 0 : \frac{t^2}{\Sigma^2(t)} \geq u\} \\ s(u) := \inf\{t > 0 : \Theta \circ a(t)t \geq u\}. \end{cases}$$

Theorem

Theorem (2017)

There exists a non-trivial càd-làg process Z_α such that

$$\left\{\frac{S_{\lfloor ut\rfloor}-\mathbf{m}_S ut}{\lambda(u)}\right\}_{t\geq 0} \quad or \quad \left\{\frac{S_{ut}-\mathbf{m}_S ut}{\lambda(u)}\right\}_{t\geq 0} \quad \stackrel{M_1}{\Longrightarrow} \quad \{Z_\alpha(t)\}_{t\geq 0}.$$

Recall on the topologies on the Skorohod space of *càd-làg* real functions on $\mathcal{D}[0,\infty)$

1. $\mathcal{C}$-topology: uniform convergence on compacts;
2. J_1-topology: uniform convergence on compacts up to small homeomorphic perturbations on the time variable;
3. M_1-topology: uniform convergence on compacts of the complete graph in $\mathbb{R}^2$.

Specifications: $\alpha \in \{1\} \sqcup (1,2) \sqcup \{2\}$

α	$a(u)$	$s(u)$	β	Top.	Z_α
2	$\Xi_2(u)\sqrt{u}$	$\frac{u}{\mathbf{d}_T}$		J_1 [a]	B
$(1,2)$	$\Xi_\alpha(u)u^{1/\alpha}$	$\frac{u}{\mathbf{d}_T}$	$\frac{(1-\mathbf{d}_S)^\alpha(1+\mathbf{b}_S)-(1+\mathbf{d}_S)^\alpha(1-\mathbf{b}_S)}{(1-\mathbf{d}_S)^\alpha(1+\mathbf{b}_S)+(1+\mathbf{d}_S)^\alpha(1-\mathbf{b}_S)}$	J_1	$\mathcal{S}_{\alpha,\beta}$
1^+	$\Xi_1(u)u$	$\frac{u}{\mathbf{d}_T}$	0	J_1	$\mathcal{C}$
1^-	$\Xi_1(u)u$	$\frac{u}{D(u)}$ [b]	0	J_1	$\mathcal{C}$

Figure: Summary of the cases for $S_{\lfloor \cdot \rfloor}$.

The limit process is Markovian !

[a] $\mathcal{C}$ for the continuous interpolation process.

[b] It corresponds to the non integrable case: D is ultimately non-decreasing and $\lim_{u\to\infty} D(u)/\Xi_1 \circ s(u) = \infty$

The anomalous case: $\alpha \in (0,1)$

Theorem (2017)

- *Suppose* $\alpha \in (0,1)$, *then the the famlly of stochastic processes* $\{S_{ut}/u\}_{t\geq 0}$ *converges in distribution for the* $\mathcal{C}$*-topology toward a self-similar (of index 1) non Markovian process* $\{\mathcal{S}_\alpha(t)\}_{t\geq 0}$.
- *The marginal* $\mathcal{S}_\alpha(t)$ *admits a density function* f_t *supported by* $(-t,t)$ *which is up to an affine transformation a generalized arcsine Lamperti distribution:*

$$f_t(x) = \frac{2\sin(\pi\alpha)(\pi t)^{-1}(t-x)^{\alpha-1}(t+x)^{\alpha-1}}{\mathbf{r}_S(t-x)^{2\alpha} + 2\cos(\pi\alpha)(t+x)^\alpha(t-x)^\alpha + \mathbf{r}_S^{-1}(t+x)^{2\alpha}}$$

with $\mathbf{r}_S := (1+\mathbf{m}_S)/(1-\mathbf{m}_S)$.

- *Finally, these densities satisfy — in a weak sense — the fractional partial differential equation*

$$\left[\frac{1+\mathbf{m}_S}{2}(\partial_x+\partial_t)^\alpha + \frac{1-\mathbf{m}_S}{2}(\partial_x-\partial_t)^\alpha\right] f_t(x) = \frac{1}{\Gamma(1-\alpha)t^\alpha}\left[\frac{1+\mathbf{m}_S}{2}\delta_t(dx) + \frac{1-\mathbf{m}_S}{2}\delta_{-t}(dx)\right].$$

Integral representation of $\mathcal{S}_\alpha$

1. Let T_α be an α-stable subordinator with null drift, *i.e.* a *càd-làg* non decreasing pure jump process.
2. Denote by $\overline{\mathcal{R}_\alpha} := \overline{\{T_\alpha(t) : t \geq 0\}}$ the regenerating set.
3. Due to the null drift assumption, $\overline{\mathcal{R}_\alpha}$ is a perfect set of Lebesgue measure zero.
4. Denoting by $\mathcal{J}$ the (countable) random set of jumps, it follows

$$\overline{\mathcal{R}_\alpha}^{\complement} = \bigsqcup_{u \in \mathcal{J}} (T_\alpha(u^-), T_\alpha(u)).$$

5. Each open interval can be labelled by $\{-1, 1\}$ accordingly with a Rademacher random variable of parameter $(1 + \mathbf{b}_S)/2$. It gives rise to some process $\{\mathcal{X}(t)\}_{t \geq 0}$. Note that its values on the regenerating set is irrelevant. Then,

$$\mathcal{S}_\alpha(t) = \int_0^t \mathcal{X}(s)\, ds.$$

Why the generalized arcsine density ?

Let $\{X_n\}_{n\geq 0}$ be any random process on a state space $A \sqcup \{\sigma\} \sqcup B$ with a checkpoint σ.

- Let N_n be the occupation time, up to time n, of the set A;
- Let $F(x)$ be the generating function of the first return time of σ.

Theorem (Lamperti, 1958)

The sequence $\left\{\frac{N_n}{n}\right\}_{n\geq 1}$ *converges in distribution to a non degenerate limit if and only if there exist* $\alpha, \rho \in (0,1)$ *such that*

$$\lim_{n\to\infty} \mathbf{E}\left[\frac{N_n}{n}\right] = \rho \quad \textit{and} \quad \lim_{x\uparrow 1} \frac{(1-x)F'(x)}{1-F(x)} = \alpha.$$

Setting $\mathbf{r} := (1-\rho)/\rho$, *the density of the limiting distribution is*

$$\frac{\sin(\pi\alpha)}{\pi} \frac{t^{\alpha-1}(1-t)^{\alpha-1}}{\mathbf{r}t^{2\alpha} + 2\cos(\pi\alpha)t^{\alpha}(1-t)^{\alpha} + \mathbf{r}^{-1}(1-t)^{2\alpha}}.$$

Application to our context

We consider the 2-order letter process $X_k X_{k+1}$ living in the state space $\{\mathtt{d},\mathtt{u}\}^2$. A slight adaptation of Lamperti's result with

- $A := \{\mathtt{d}^2\}$ and $B := \{\mathtt{u}^2\}$,
- checkpoints $\sigma := \{\mathtt{ud}\}$ and $\eta := \{\mathtt{d},\mathtt{u}\}$,

implies our result.

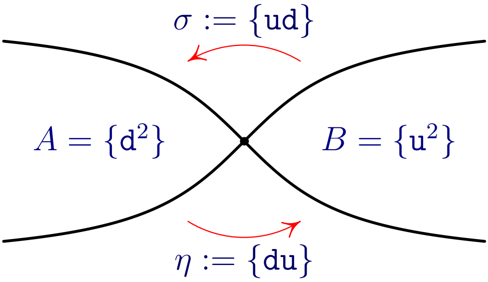

Figure: Lamperti's Theorem with directed checkpoints.

The standard case: $\alpha \in (1, 2]$

We set $N(t) := \inf\{n \geq 0 : T_{n+1} > t\}$ and take advantage of the decomposition

$$S_{\lfloor ut \rfloor} - \mathbf{m}_S ut = \left[M_{N(ut)} - \mathbf{m}_S T_{N(ut)}\right] + R(ut) := C_{N(ut)} + R(ut),$$

where $\{C_n\}_{n\geq 0}$ denotes the RW with jumps distributed as $\tau_1^{\mathtt{c}}$ and $\{R(\nu)\}_{\nu \geq 0}$ is the residual process satisfying

$$|R(\nu)| \leq (1 - \mathbf{m}_S)\tau^{\mathtt{u}}_{N(\nu)+1} + (1 + \mathbf{m}_S)\tau^{\mathtt{d}}_{N(\nu)+1}.$$

Step 1 Standard arguments about processes with stationary and independent increments implies C belongs to the domain of attraction of the α-stable Lévy process $\mathcal{S}_{\alpha,\beta}$ — tightness.

Step 2 WLLN applied to T

$$\left\{\frac{N(ut)}{s(u)}\right\}_{t\geq 0} \longrightarrow \{t\}_{t\geq 0}.$$

Step 3 Residual term converges to the null process in probability and a J_1 continuous mapping argument implies the convergence of finite dimensional distribution.

The case $\alpha = 1$ and a word on tightness

We need to distinguish the case $\alpha = 1^+$ and $\alpha = 1^-$:

- in case $\alpha = 1^+$, τ^ℓ are both integrable, but the step 1 is no longer true. It is needed to look carefully at the centering term: replace $\mathbf{m}_S$ with $\mathbf{b}_S$.
- in case $\alpha = 1^-$, the non-integrable case, the step 2 is no longer obvious because of the use of the WLLN.

NB: For $\alpha \in (0,1)$, the modulus of continuity of $\{S_{ut}/u\}_{t\geq 0}$ is 1 a-s. so that the tightness is obvious.