Identification and isotropy characterization of deformed random fields through excursion sets

Julie Fournier

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MAP5, université Paris Descartes
Work supervised by Anne Estrade

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The deformed random fields model

- Let $X : \mathbb{R}^2 \to \mathbb{R}$ be a **stationary and isotropic random field**:

  for any translation τ, for any rotation ρ in $\mathbb{R}^2$,

  $$X \circ \tau \overset{\text{law}}{=} X \quad \text{and} \quad X \circ \rho \overset{\text{law}}{=} X.$$

  We write $C(t) = \mathrm{Cov}(X(t), X(0)) = \mathrm{Cov}(X(s+t), X(s))$ its covariance function. We call X the **underlying field**.

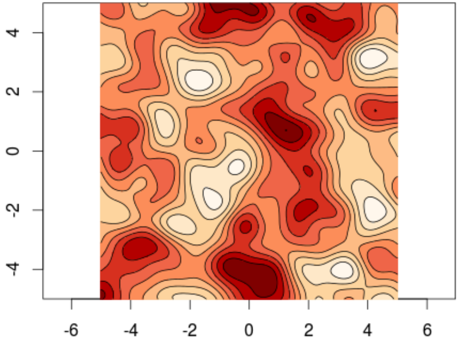
- let $\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be a bijective, bicontinuous, deterministic application satisfying $\theta(0) = 0$, which we will call a **deformation**.

$X_\theta = X \circ \theta : \mathbb{R}^2 \to \mathbb{R}$ is the **deformed random field** constructed with the underlying field X and the deformation θ.

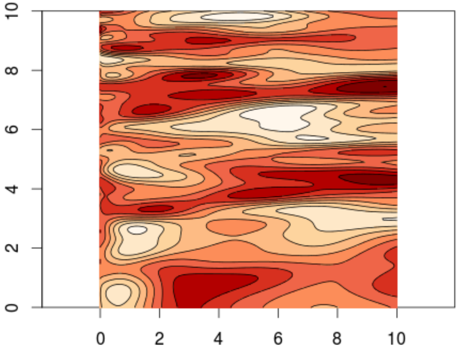
Two types of question :

- invariance properties of the deformed field
- inverse problem: identification of θ thanks to observations of X_θ.

First observation: the invariance properties are not preserved in general.



Level sets of a realization of a Gaussian stationary and isotropic random field X with Gaussian covariance $C(x) = \exp(-\|x\|^2)$.



Level sets of a realization of X_θ constructed with $\theta : (s,t) \mapsto (s^{0.6}, t^{1.4})$ and with the underlying field X.

Question : which are the deformations that preserve stationarity and isotropy ?

References

- Spatial statistics (Sampson and Guttorp, 1992).
- Image analysis : "shape from texture" issue (Clerc-Mallat, 2002)
- Numerous domains of application in physics:
  for instance, used in cosmology for the modelization of the CMB.
- One possible angle of study : study of the covariance function of X_θ :

$$\boxed{C_\theta(x,y) = \mathrm{Cov}(X_\theta(x), X_\theta(y)) = C(\theta(x) - \theta(y))}$$

  (Perrin-Meiring, 1999; Perrin-Senoussi, 2000).
- Problem of the estimation of θ, up to rotation and translation:

$$\text{if } \rho \in SO(2) \text{ and } a \in R^2 \text{ then } X_{\rho\circ\theta+a} \overset{\text{law}}{=} X_\theta .$$

- Particular case of the model of a deterministic deformation operator applied to a random field satisfying invariance properties: $Y = DX$ (Clerc-Mallat, 2003).

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Assumptions

The **underlying field** X must satisfy the following assumptions :

$$\textbf{(H)} \begin{cases} X \text{ is stationary and isotropic,} \\ X \text{ is centered and admits a second moment.} \end{cases}$$

The **deformation** θ belongs to the set

$$\mathcal{D}^0(\mathbb{R}^2) = \{\theta : \mathbb{R}^2 \to \mathbb{R}^2 \,/\, \theta \text{ is continous and bijective, with a continuous inverse, such that } \theta(0) = 0\}$$

Cases of isotropy (1)

Problem

Which are the deformations θ such that **for any underlying random field $\mathbf{X}$**, X_θ is isotropic ?

- **A different problem** : Which are the deformations θ such that for a fixed underlying random field X, X_θ is isotropic ?
- **Example :** elements of SO(2) : rotations of $\mathbb{R}^2$.
- **Elements of proof.**
  - Invariance of the covariance function of X_θ under rotations :

$$\forall \rho \in SO(2), \forall (x,y) \in (\mathbb{R}^2)^2,$$
$$C(\theta(\rho(x)) - \theta(\rho(y))) = C(\theta(x) - \theta(y))$$

  - Chose the covariance function $C(x) = \exp(-\|x\|^2)$ to obtain

$$\forall \rho \in SO(2),\ \forall (x,y) \in (\mathbb{R}^2)^2, \quad \|\theta(\rho(x)) - \theta(\rho(y))\| = \|\theta(x) - \theta(y)\|.$$

  - Polar representation of θ.

Cases of isotropy (2)

Notations : $\hat{\theta}$ polar representation of θ :
$\hat{\theta} : (0,+\infty) \times \mathbb{Z}/2\pi\mathbb{Z} \to (0,+\infty) \times \mathbb{Z}/2\pi\mathbb{Z} \quad (r,\varphi) \mapsto (\hat{\theta}_1(r,\varphi), \hat{\theta}_2(r,\varphi))$.

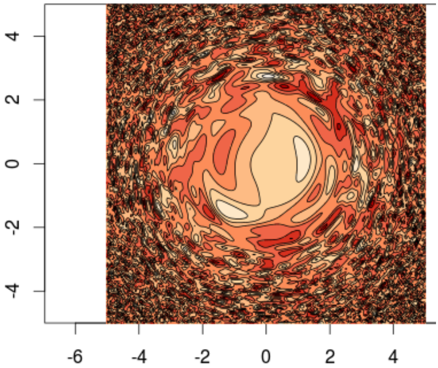
Definition

A deformation $\theta \in \mathcal{D}^0(\mathbb{R}^2)$ is a spiral deformation if there exist $f : (0,+\infty) \to (0,+\infty)$ strictly increasing and surjective, $g : (0,+\infty) \to \mathbb{Z}/2\pi\mathbb{Z}$ and $\varepsilon \in \{\pm 1\}$ such that θ satisfies

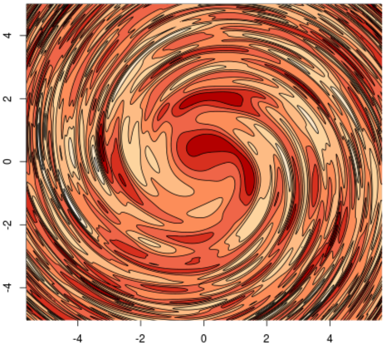
$$\forall (r,\varphi) \in (0,+\infty) \times \mathbb{Z}/2\pi\mathbb{Z}, \quad \hat{\theta}(r,\varphi) = (f(r),\, g(r) + \varepsilon\varphi).$$

Answer to the problem

Spiral deformations are the deformations making X_θ isotropic for any underlying field X.



Level sets of a realization of X_θ with a deformation $\theta : x \mapsto \|x\|\, x$ and with X Gaussian with Gaussian covariance.



Level sets of a realization of X_θ with a deformation with polar representation $\hat{\theta} : (r, \varphi) \mapsto (\sqrt{r}, r + \varphi)$ and a Gaussian underlying field with Gaussian covariance.

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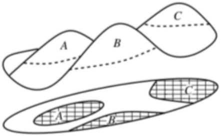
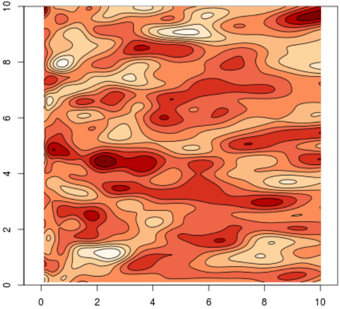
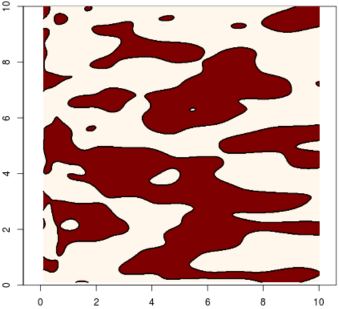
Excursion sets

- Let $u \in \mathbb{R}$ be a fixed level,
- let T be a rectangle or a segment in $\mathbb{R}^2$,
- let $A_u(X_\theta, T)$ be the **excursion set** of X_θ restricted to T above level u:

$$A_u(X_\theta, T) = \{t \in T \,/\, X_\theta(t) \geq u\}$$







Level sets and excursion sets of a realization of X_θ, with $\theta : (s,t) \mapsto (s^{0.6}, t)$ defined on $(0, +\infty)^2$ and X Gaussian with Gaussian covariance.

Euler characteristic χ of excursion sets

Euler characteristic: integer-valued and additive functional defined on a large class of compact sets.

Heuristic definition for a compact set $G \subset \mathbb{R}^2$ of dimension 1 or 2

- $d = 1$, $\chi(G) = \#(\text{disjoint components in G})$;
- $d = 2$, $\chi(G) = \#(\text{connected components in G}) - \#(\text{holes in G})$.

The **Euler characteristic** χ is a homotopy invariant and $A_u(X_\theta, T) = \theta^{-1}(A_u(X, \theta(T))$, hence

$$\boxed{\chi(A_u(X_\theta, T)) = \chi(A_u(X, \theta(T)))}.$$

and we can use an expectation formula proven for stationary and isotropic random fields in Adler-Taylor, 2007.

Additional assumptions

(H')
$$\begin{cases} \textbf{X is Gaussian} : \\ \qquad \forall n \in \mathbb{N},\ \forall (t_i)_{1 \leq i \leq n} \in (\mathbb{R}^2)^n, \quad (X(t_i))_{1 \leq i \leq n} \text{ is a Gaussian vector,} \\ X \text{ is stationary and isotropic,} \\ \textbf{X is almost surely of class } \mathcal{C}^{\mathbf{2}}, \\ X \text{ is centered, } C(0) = 1 \text{ and } C''(0) = -I_2, \\ \textbf{a non-degeneracy assumption on X(t), for every t} \in \mathbb{R}^{\mathbf{2}}. \end{cases}$$

The deformation θ belongs to the set

$$\mathcal{D}^2(\mathbb{R}^2) = \{\theta : \mathbb{R}^2 \to \mathbb{R}^2 \,/\, \theta \textbf{ of class } \mathcal{C}^{\mathbf{2}} \text{ and bijective,} \\ \textbf{with an inverse of class } \mathcal{C}^{\mathbf{2}}, \\ \text{such that } \theta(0) = 0\}$$

Formulas for the expectation of $\mathbb{E}[\chi(A_u(X_\theta, T))]$ (Adler, Taylor (2007))

• If T is a segment in $\mathbb{R}^2$, writing $|\theta(T)|_1$ the one-dimensional Hausdorff measure of $\theta(T)$,

$$\boxed{\mathbb{E}[\chi(A_u(X_\theta, T))] = e^{-u^2/2} \frac{|\theta(T)|_1}{2\pi} + \Psi(u)},$$

where $\Psi(u) = \mathbb{P}(Y > u)$ for $Y \sim \mathcal{N}(0,1)$.

• If $T \subset \mathbb{R}^2$ is a rectangle, writing $|\theta(T)|_2$ the two-dimensional Hausdorff measure of $\theta(T)$,

$$\boxed{\mathbb{E}[\chi(A_u(X_\theta, T))] = e^{-u^2/2} \left(u \frac{|\theta(T)|_2}{(2\pi)^{3/2}} + \frac{|\partial\theta(T)|_1}{4\pi} \right) + \Psi(u)},$$

where ∂G is the frontier of G.

Writing $\theta = (\theta_1, \theta_2)$ the coordinate functions of θ, let $J_\theta(s,t)$ be the **Jacobian matrix** of θ at point $(s,t) \in \mathbb{R}^2$:

$$J_\theta(s,t) = \begin{pmatrix} \frac{\partial \theta_1}{\partial s}(s,t) & \frac{\partial \theta_1}{\partial t}(s,t) \\ \frac{\partial \theta_2}{\partial s}(s,t) & \frac{\partial \theta_2}{\partial t}(s,t) \end{pmatrix} = \begin{pmatrix} J_\theta^1(s,t) & J_\theta^2(s,t) \end{pmatrix}.$$

Note that the determinant of $J_\theta(x)$ is either positive on $\mathbb{R}^2$ or negative on $\mathbb{R}^2$.

- $|\theta([0,s] \times [0,t])|_2 = \int_0^s \int_0^t |\det(J_\theta(x,y))| \, dx \, dy$
- $|\theta([0,s] \times \{t\})|_1 = \int_0^s \|J_\theta^1(x,t)\| \, dx$
- $|\theta(\{s\} \times [0,t])|_1 = \int_0^t \|J_\theta^2(s,y)\| \, dy$

Consequence : general idea

Condition / information on $\mathbb{E}[\chi(A_u(X, \theta(T)))]$ (T rectangle or segment) implies condition / information on the Jacobian matrix of θ, hence on θ.

A weak notion of isotropy linked to excursion sets

Let X be an underlying field satisfying **(H')**.

Definition (χ-isotropic deformation)

A deformation $\theta \in \mathcal{D}^2(\mathbb{R}^2)$ is χ-isotropic if for any rectangle T in $\mathbb{R}^2$, for any $u \in \mathbb{R}$ and for any $\rho \in SO(2)$,

$$\mathbb{E}[\chi(A_u(X_\theta, \rho(T))] = \mathbb{E}[\chi(A_u(X_\theta, T)].$$

- Definition depending on the underlying field X.
- Same definition can be stated with a fixed level $u \neq 0$.
- Remark: θ spiral deformation $\Rightarrow$ θ χ-isotropic deformation.
- Therefore, if θ is χ-isotropic, X_θ can be considered as **weakly isotropic**.

Aim : Prove that

$$\theta \ \chi\text{-isotropic deformation} \quad \Rightarrow \quad \theta \text{ spiral deformation.}$$

First characterization

Elements of proof

- The χ-isotropic condition is also true for T segment, thanks to a continuity property.
- Formulas for $\mathbb{E}[\chi(A_u(X_\theta, T)]$ involve J_θ,
  formulas for $\mathbb{E}[\chi(A_u(X_\theta, \rho(T))]$ involve $J_{\theta\circ\rho}$.

Lemma 1

A deformation $\theta \in \mathcal{D}^2(\mathbb{R}^2)$ is χ-isotropic if and only if for any $\rho \in SO(2)$, for any $x \in \mathbb{R}^2$,

$$\begin{cases} (i) & \forall k \in \{1,2\},\ \|J^k_{\theta\circ\rho}(x)\| = \|J^k_\theta(x)\|, \\ (ii) & \det(J_{\theta\circ\rho}(x)) = \det(J_\theta(x)). \end{cases}$$

Second characterization and conclusion of the proof

A translation of the first lemma in polar coordinates brings:

Lemma 2

A deformation $\theta \in \mathcal{D}^2(\mathbb{R}^2)$ is a χ-isotropic deformation if and only if functions

$$\begin{cases} (r,\varphi) \mapsto (\partial_r \hat{\theta}_1(r,\varphi))^2 + (\hat{\theta}_1(r,\varphi)\,\partial_r \hat{\theta}_2(r,\varphi))^2 \\ (r,\varphi) \mapsto (\partial_\varphi \hat{\theta}_1(r,\varphi))^2 + (\hat{\theta}_1(r,\varphi)\,\partial_\varphi \hat{\theta}_2(r,\varphi))^2 \\ (r,\varphi) \mapsto \hat{\theta}_1(r,\varphi)\,\det(J_{\hat{\theta}}(r,\varphi)) \end{cases}$$

are radial, i.e. if they do not depend on φ.

This differential system is solved in Briant, F.(2017, submitted) and **the set of solutions is exactly the set of spiral deformations**.

Chain of equalities

We write

- $\mathcal{S}$ the set of spiral deformations in $\mathcal{D}^2(\mathbb{R}^2)$,
- $\mathcal{I}$ the set of deformations $\theta \in \mathcal{D}^2(\mathbb{R}^2)$ such that **for any underlying field** X satisfying $(\mathbf{H}')$, X_θ is isotropic,
- for a **fixed** underlying field X satisfying $(\mathbf{H}')$,

$$\mathcal{I}(X) = \{\theta \in \mathcal{D}^2(\mathbb{R}^2) \text{ such that } X_\theta \text{ is isotropic}\}.$$

- $\mathcal{X}$ the set of χ-isotropic deformations.

Corollary

Let X be a stationary and isotropic random field satisfying $(\mathbf{H}')$. Then $\mathcal{S} = \mathcal{I} = \mathcal{I}(X) = \mathcal{X}$.

Conclusion : For deformed random fields, a weak notion of isotropy based on excursion sets coincides with isotropy in law.

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Existing methods

We assume that θ is unknown.

Different methods have been studied to identify θ thanks to the observation of X_θ:

- use several observations of whole realizations of X_θ on a sparse grid (Sampson-Guttorp, 1992)
- use only one observation of a whole realization of X_θ but on a dense grid (Guyon-Perrin, 2000, Clerc-Mallat, 2003, Anderes-Stein, 2008 ...)
- use sparse observation(s) of X_θ : level curves (Cabaña, 1987) or excursion sets (our method).

Our method

We use the information provided by $\mathbb{E}[\chi(\mathbf{A_u}(\mathbf{X}_\theta, \mathbf{T}))]$, for a fixed $\mathbf{u} \neq \mathbf{0}$.

In the following, we assume that θ is unknown but that $\mathbb{E}[\chi(A_u(X_\theta, T))]$ is known for T rectangle or segment in $\mathbb{R}^2$.

We in fact use a modified version of χ and we assume that $\forall x \in \mathbb{R}^2$, $\det(J_\theta(x)) > 0$.

Problem of the estimation of $\mathbb{E}[\chi(\mathbf{A_u}(\mathbf{X}_\theta, \mathbf{T}))]$

- If X_θ is not stationary nor isotropic, we need several realizations of X_θ on a fixed domain.
- If X_θ is **stationary**, one realization is enough.
- If X_θ is **isotropic**, we propose an estimator based on the χ-isotropy condition, computed thanks to one realization.

Identification of θ thanks to excursions sets of X_θ (1)

Linear case : $\theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}$. $u \neq 0$. Three excursion sets above

$$T = [0,s] \times \{0\},\ T = \{0\} \times [0,t],\ T = [0,s] \times [0,t], \text{ with } (s,t) \in (\mathbb{R}^*)^2$$

allow to compute

$$a = \sqrt{\theta_{11}^2 + \theta_{21}^2}, \quad b = \sqrt{\theta_{12}^2 + \theta_{22}^2} \quad \text{and} \quad c = \theta_{11}\theta_{22} - \theta_{21}\theta_{12}.$$

Therefore, there exists $(\alpha, \beta) \in (\mathbb{Z}/2\pi\mathbb{Z})^2$ such that

$$\theta = \begin{pmatrix} a\cos(\alpha) & b\cos(\beta) \\ a\sin(\alpha) & b\sin(\beta) \end{pmatrix} = \rho_\alpha \begin{pmatrix} a & b\cos(\delta) \\ 0 & b\sin(\delta), \end{pmatrix},$$

with $\delta = \beta - \alpha$ satisfying $c = ab\sin(\delta)$. Consequently, θ belongs to the set

$$\mathcal{M}(a,b,c) = \left\{ \rho \begin{pmatrix} a & \sqrt{b^2 - (ca^{-1})^2} \\ 0 & ca^{-1} \end{pmatrix},\ \rho \begin{pmatrix} a & -\sqrt{b^2 - (ca^{-1})^2} \\ 0 & ca^{-1} \end{pmatrix},\ \rho \in SO(2) \right\}.$$

Identification of θ thanks to excursions sets of X_θ (2)

General case. (We add some assumptions on θ.)

- For any $x \in \mathbb{R}^2$, writing $J_\theta(x) = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}$, we apply the results above to conclude that $J_\theta(x) \in \mathcal{M}(a, b, c)$ (now depending on x).
- Consequently, the complex dilatation $\mu = \frac{\partial_{\bar{z}}\theta}{\partial_z \theta}$ at point x can be determined, up to complex conjugation, in fonction of a, b and c.
- The mapping theorem formulates a characterization of a deformation up to a conformal mapping through its complex dilatation μ.

Thanks for your attention !

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